REVIEW

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Different solutions of the diffusion equation and its applications

Khaled S. M. Essa^{1*} , Soad M. Etman¹, Maha S. El-Otaify¹, M. Embaby¹, Ahmed M. Mosallem¹ and Ahmed S. Shalaby²

Abstract

In this report, we solved the advection–diffusion equation under pollutants deposition on the ground surface, taking wind speed and vertical diffusion depend on the vertical height. Also, we estimated a simple diffusion model from point source in an urban atmosphere and the conservative material with downwind was evaluated. Then, we calculated the extreme ground-level concentration as a function of stack height and plume rise in two cases. Comparison between the proposed models and the emission from the Egyptian Atomic Research Reactor at Inshas had been done. Lastly, we discussed the results in this report.

Keywords: Advection–diffusion equation, Pollutants deposition, Extreme ground-level concentration, Egyptian atomic research reactor, Urban atmosphere, The conservative material, Continuous point source, Maximum concentration value, The logarithmic law of wind velocity, Contaminants in the planetary boundary layer

1 Background

The Gaussian plume equation was semi-analytical solution, assuming that wind velocity and eddy diffusivities were constant. The non-Gaussian models agree well with the observed data by Hinrichsen [17]. Wortmann et al. [35] and Essa et al. [10] calculated a new technique to get the concentration of contaminants in the planetary boundary layer.

Early solutions used only two dimensions [26, 28, 34]. The Green's function method was studied by Stakgold [30]. Yeh [37] calculated advection–diffusion equation in three dimensions under boundary conditions. Also, these equations have been studied in turbulent models for unbounded domain by [18, 38]. Also, [33, 32] utilized dispersion modeling. All techniques have been restricted to a single isolated point source located at the origin. The details of several solutions were investigated by Carslaw and Jaeger [7], Sutton [31], Yaglom [36], Pasquill [24],

Berlyand [4] and Lin and Hildemann [21]. Also, the diffusion from a point source in an urban atmosphere was estimated by Essa and El-Otaify [9]. Heines and Peters [16], and Bennett [3] investigated the physical effect of model for the dry deposition of pollutants to a rough surface. Recently, Essa et al. [11, 12] estimated the solution of the advection–diffusion equation in two dimensions with variable vertical eddy diffusivity and wind speed using Hankel transform. Also, Essa et al. [11, 12] found the solution of advection–diffusion equation in three dimensions using Hankel transform.

In the first part: Pasquill and Smith [22, 23] introduced the advection–diffusion equation in steady state for a continuous point source as follows:

$$u(z) = \frac{\partial C}{\partial x} = \frac{\partial}{\partial z} \left[K(z) \frac{\partial C}{\partial z} \right] \quad (1)$$

where $u(z)$ and $K(z)$ are the mean horizontal wind velocity and vertical eddy diffusivity as a function of vertical height. Assuming that $u(z) = 0$ at $z = 0$, as defined by [29] as follows:

*Correspondence: mohamedksm56@yahoo.com

¹ Mathematics and Theoretical Physics Department, NRC, Egyptian Atomic Energy Authority, Cairo, Egypt
Full list of author information is available at the end of the article

$$\frac{du}{dz} = \frac{u_*}{kz} \phi_m(z/L) \quad (2)$$

where $\phi_m(z/L)$ is the non-dimension wind gradient, L is the Monin–Obukhov length, u_* is the friction velocity, and k is the von-Karman's constant equals 0.4. One can get by integrating Eq. (2) w.r.t (z) the following velocity profiles [9]:

$$u(z) = \frac{u_*}{k} \ln\left(\frac{z+z_o}{z_o}\right), \text{ neutral conditions} \quad (3)$$

$$u(z) = \frac{u_*}{k} \left[\ln\left(\frac{z+z_o}{z_o}\right) + \frac{5.2z}{L} \right], \text{ stable conditions} \quad (4)$$

$$u(z) = \frac{u_*}{k} \left\{ \begin{aligned} &\ln \left[\frac{[1 + \frac{16(z+z_o)}{L}]^{1/4} - 1}{[1 + \frac{16(z+z_o)}{L}]^{1/4} + 1} \right] + 2 \tan^{-1} \left[1 + \frac{16(z+z_o)}{L} \right]^{1/4} \\ &+ \ln \left[\frac{[1 + \frac{16z_o}{L}]^{1/4} + 1}{[1 + \frac{16z_o}{L}]^{1/4} - 1} \right] + 2 \tan^{-1} \left[1 + \frac{16z_o}{L} \right]^{1/4} \end{aligned} \right\} \text{unstable conditions} \quad (5)$$

The vertical diffusion coefficient was taken from Hanna et al. [13, 14] as follows:

$$K(z) = K_o + k u_* z \quad \text{neutral conditions} \quad (6)$$

$$K(z) = K_o + k u_* z \left(1 + \frac{5.2z}{L} \right)^{-1} \quad \text{stable conditions} \quad (7)$$

$$K(z) = K_o + k u_* z \left(1 + \frac{16z}{|L|} \right)^{1/4} \quad \text{unstable conditions} \quad (8)$$

where K_o is a constant value. Equation (1) is solved using the boundary conditions as follows:

(1)-The mixing layer was assumed not allowing fluid to pass through to the pollutants

$$C(x, z) = 0, \quad \text{at } z = h \quad (9a)$$

And

$$\frac{\partial C(x, z)}{\partial z} = 0 \quad \text{at } z = h \quad (9b)$$

where h is the height of the planetary boundary layer (PBL).

(2)-Assuming that there is a deposition at the ground surface as follows:

$$K(z) \frac{\partial C(x, z)}{\partial z} = v_d C(x, z) \quad \text{at } z = 0 \quad (10)$$

where v_d is the deposition velocity.

The solution of Eq. (1) is solved theoretical under different stabilities [10] as follows:

$$C(x, z) = F(x) \left(1 - \frac{z}{h} \right)^2 \quad (11)$$

Integrating Eq. (1) from zero to h with respect to z , and applying Eqs. (9b) and (10), one gets:

$$\frac{d}{dx} \int_0^h u(z) C(x, z) dz = -v_d C(x, 0) \quad (12)$$

2 For neutral conditions

Substituting $C(x, z)$ and $u(z)$ by Eqs. (11) and (3) in Eq. (12), one obtains:

$$\frac{u_*}{k} \frac{dF(x)}{dx} \int_0^h \left(1 - \frac{z}{h} \right)^2 \ln \left(\frac{z+z_o}{z_o} \right) dz = -v_d F(x) \quad (13)$$

Equation (13) becomes:

$$\frac{u_* N}{k} \frac{dF(x)}{dx} = -v_d F(x) \quad (14)$$

where

$$\begin{aligned} N &= \int_0^h \left(1 - \frac{z}{h} \right)^2 \ln \left(\frac{z+z_o}{z_o} \right) dz \\ &= \frac{z_o^3}{3h^2} \left(1 + \frac{h}{z_o} \right)^3 \ln \left(1 + \frac{h}{z_o} \right) \\ &\quad - \frac{z_o^2}{18h} \left[\left\{ 6 + \frac{15}{z_o} + 11 \frac{h^2}{z_o^2} \right\} \right] \end{aligned} \quad (15)$$

Equation (14) becomes:

$$F(x) = F_0 \exp \left(- \frac{v_d k}{u_* N} x \right) \quad (16)$$

where F_0 is a constant. Then, the solution in the **neutral** case was obtained as follows:

$$C(x, z) = F_0 \exp\left(-\frac{v_d k}{u_* N} x\right) \left(1 - \frac{z}{h}\right)^2 \quad (17)$$

Taking

$$x_{dn} = \frac{u_* N}{v_d k} \quad (18)$$

where x_{dn} is the decay distance of pollutant in neutral condition. Equation (17) becomes:

$$C(x, z) = F_0 \exp\left(-\frac{x}{x_{dn}}\right) \left(1 - \frac{z}{h}\right)^2 \quad (19)$$

Now, F_0 can be determined by using the formula:

$$Q = \int_{x=0}^{x_{dn}} \int_{z=0}^h u(z) C(x, z) dx dz \quad (20)$$

where Q is the emission rate which after substitution of Eqs. (3) and (19), one gets:

$$\begin{aligned} Q &= \frac{u_* F_0}{k} \int_{x=0}^{x_{dn}} \exp\left(-\frac{x}{x_{dn}}\right) dx \int_{z=0}^h \left(1 - \frac{z}{h}\right)^2 \ln\left(\frac{z+z_o}{z_o}\right) dz \\ &= \frac{u_* F_0 N x_{dn}}{k} \left(1 - \frac{1}{e}\right) \end{aligned}$$

Then, the concentration of pollutants in **neutral** condition becomes:

$$C(x, z) = \frac{Q}{0.63 v_d x_{dn}^2} \exp\left(-\frac{x}{x_{dn}}\right) \left(1 - \frac{z}{h}\right)^2 \quad (21)$$

Similarly, $C(x, z)$ in **stable** condition takes form [10]:

$$C(x, z) = \frac{Q}{0.63 v_d x_{ds}^2} \exp\left(-\frac{x}{x_{ds}}\right) \left(1 - \frac{z}{h}\right)^2 \quad (22)$$

where x_{ds} is given in the stable case as follows:

$$x_{ds} = \frac{u_* M}{v_d k} \quad (23)$$

where $M = N - (5.2 h^2/12L)$.

Also, $C(x, z)$ in **unstable** condition is given [10] as follows:

$$C(x, z) = \frac{Q}{0.63 v_d x_{du}^2} \exp\left(-\frac{x}{x_{du}}\right) \left(1 - \frac{z}{h}\right)^2 \quad (24)$$

where x_{du} is given in unstable case as follows:

$$x_{du} = \frac{u_* D}{v_d k} \quad (25)$$

where D is taken from [10].

In the second part: We assumed two-dimensional structure with homogeneity in the lateral coordinate, where $\bar{u}(z)$ is described in Fig. 1. The ground surface is treated with no deposition of matter exist.

The conservation of mass can be taken from [9]:

$$Q = \int_0^H \bar{u}(z)C(z)dz \quad (26)$$

where

$\bar{u}(z)$ is the average wind velocity, $C(z)$ is the concentration of pollutants, and H is the effective stack height of the plume.

3 The Effective stack Height (H)

We estimated the height of the plume as follows:

$$\Delta h = 3(w_o/u)D_1 \quad (27)$$

where w_o is the exit velocity of the pollutants (m/s) and D_1 is the inside stack diameter (m). Then, H equals:

$$H = h_s + \Delta h = h_s + 3(w_o/u)D_1 \quad (28)$$

The concentration profile will be assumed in form [9]:

$$C_0 = \frac{kHQ}{v_*} \left\{ \frac{1}{4H} \left[2(H+z_0)(2H+\alpha H-\alpha z_0) \ln \frac{H+z_0}{z_0} - H(4H+\alpha H-2\alpha z_0) \right] + \frac{5.2H^2}{L} \left(\frac{1}{2} + \frac{\alpha}{3} \right) \right\}^{-1} \quad (34)$$

5 Stable case

$$Q = \int_0^H \left[\frac{v_*}{k} \ln \left(\frac{z+z_0}{z_0} \right) \right] C_0(1+\alpha(z/H))dz \quad (31)$$

The integration over "z" yields:

$$C_0 = \frac{4kHQ}{v_*} [2(H+z_0)(2H+\alpha H-\alpha z_0) \ln \frac{H+z_0}{z_0} - H(4H+\alpha H-2\alpha z_0)]^{-1} \quad (32)$$

$$Q = \int_0^H \left[\frac{v_*}{k} \ln \left(\frac{z+z_0}{z_0} \right) + \frac{5.2z}{L} \right] C_0(1+\alpha(z/H))dz \quad (33)$$

After integration, C_o takes the form:

6 Unstable case

$$Q = \int_0^H \frac{v_*}{k} \left[\ln \left(\frac{(1+f(z))^{\frac{1}{4}} - 1}{(1+f(z))^{\frac{1}{4}} + 1} \right) + 2 \tan^{-1} (1+f(z))^{\frac{1}{4}} + \ln \left(\frac{(1+\frac{16z_0}{L})^{\frac{1}{4}} + 1}{(1+\frac{16z_0}{L})^{\frac{1}{4}} - 1} \right) + 2 \tan^{-1} \left(1 + \frac{16z_0}{L} \right)^{\frac{1}{4}} \right] C_0(1+\alpha(z/H))dz \quad (35)$$

$$C/C_0 = 1 + \alpha_1(z/H) + \alpha_2(z/H)^2 + \dots \quad (29)$$

where C_o is the concentration value at the edge of the plume. $C(z)$ is the concentration at vertical height; α_1 and α_2 , etc., are constants.

The fitting in Fig. 2 is obtained by taking the first two terms as follows:

$$C/C_0 = 1 + \alpha_1(z/H) \quad (30)$$

If α_1 is a r percentage of the concentration as follows:

$$\alpha_1 = 0.01 r - 1 \quad (30a)$$

if $r=0$, then

$$C/C_o = 1 - (z/H) \quad (30b)$$

After substituting, Eq. (26) becomes using different stabilities as follows:

4 Neutral case

where $f(z)$ stands for: $f(z) = \frac{16(z+z_0)}{L}$.

By substituting from the above equation into Eq. (35) and integrating w.r.t "z," one gets:

$$C_0 = \frac{kQ}{v_*} \Omega^{-1}$$

where Ω was taking from [9]:

The time dosage integral of the concentration "D" is defined as:

$$D = \int_0^t C dt \quad (36)$$

The mass balance condition takes the form:

$$Q = \int_0^z \int_0^t u C dz dt = \text{constant} \quad (37)$$

Substituting from Eq. (36) in Eq. (37), one gets:

$$Q = \int_0^z u D dz = \text{constant} \quad (38)$$

The second research reactor in E.A.E.A., (ETRR-2) has stack height H_1 equals 27 m which is a point source of (I^{131}), (H) equals 31.29 m, (Q) equals 35 Bq, the wind velocity (u) equals 2.8 m/s, and the lapse rate ($\Delta T/\Delta Z$) ($^{\circ}C/100$ m) is 0.36, which is a **stable** case. Using Eq. (34), one gets (C_o) equals 1.208 Bq/m³. Then, the concentration at ground level becomes:

$$C(\text{ground}) = 1.208 \left(1 - \frac{H_1}{H} \right) = 0.17 \text{ Bq/m}^3 \quad (39)$$

The observed concentration at $x=300$ m and $H=31.29$ m was 0.16 Bq/m³. The corrected source strength becomes:

$$Q(\text{corrected}) = \frac{(0.16)(35)}{0.17} = 37.06 \text{ Bq}$$

Then, the corrected $C(\text{ground})=0.18$ Bq/m³.

In the third part: The maximum ground-level concentration with two cases was estimated [9]. The concentration at the ground reaches a maximum and thereafter decreases as given by Hans et al.[15] with the formula:

$$C(x, y) = [Q/(\pi \sigma_y \sigma_z u)] \exp \left[-H^2/2\sigma_z^2 \right] \exp \left[-y^2/2\sigma_y^2 \right] \quad (40)$$

σ_i ($i=y, z$): are dispersion parameters of concentration in the lateral and vertical directions (m).

The extreme concentration at ground occurs along the plume centerline ($y=0$) at ground ($z=0$) [25].

σ_y and σ_z can be estimated as follows:

$$\sigma_y = ax^b, \quad \sigma_z = cx^d \quad (41)$$

where a, b, c, d values are taken from Curtiss [8].

The ground-level concentration along centerline was estimated from Eq. (40) taking σ_y and σ_z from Eq. (41) one gets:

$$\chi(x, 0, 0; H) = \frac{Q}{\pi acux^{b+d}} \exp \left[\frac{-H^2(x)}{2c^2x^{2d}} \right] \quad (42)$$

The maximum concentration value can be obtained after solving the following equation:

$$xH \frac{\partial H}{\partial x} - H^2d + c^2x^{2d}(b+d) = 0 \quad (43)$$

The maximum concentration and effective height were obtained as follows:

$$\chi^*(x, 0, 0, H) = [Q/(\pi \sigma_y \sigma_z u_h)] \exp \left[\frac{-1}{2\sigma_z^2} \left[\frac{c^2x^{2d}(b+d)}{0.5 \left(\frac{1.6F^{1/3}}{u_h} \right) x^{2/3} + h_s d} \right]^2 \right] \quad (52)$$

7 First: the effective height as a function of x

A—Unstable or neutral conditions:

The buoyancy flux parameter, F_b , was written [5] as follows:

$$F_b = g v_s r_s^2 (T_s - T_a) / T_s \quad (44)$$

For ($h_s < 305$ m), g is the acceleration (ms^{-2}), v_s is the exit vertical speed (ms^{-1}), r_s is the exit radius (m), T_s is the gas exit temperature (K), and T_a is the ambient temperature (K) at h_s [27].

The critical x^* is given by:

$$x^* = 2.16 F_b^{2/5} h_s^{3/5} \quad (45)$$

For $x \leq x^*$, we have:

$$\Delta h(x) = \text{const. } F_b^{1/3} (u)^{-1} x^{2/3} \quad (46)$$

Taking the constant equals 1.6 [6], one gets:

$$H = h_s + 1.6 F_b^{1/3} (u)^{-1} x^{2/3} \quad (47)$$

This equation is used if $T_s > T_a$ (2/3 law) [27].

u_h can be calculated at 10 m as follows:

$$u_h = u_{10} (H/10m)^p \quad (48)$$

where the parameter p is given from [20].

Substituting from (47) into (43), one gets:

$$H^* = \frac{-c^2 x^{2d} (b+d)}{2/3 \left(\frac{1.6F^{1/3}}{u_h} \right) x^{2/3} - d \left(\frac{1.6F^{1/3}}{u_h} \right) x^{2/3} - h_s d} \quad (49)$$

where H^* is the maximum value.

In unstable condition, taking $d = 1.17$ ([8] H^* becomes:

$$H^* = \frac{c^2 x^{2d} (b+d)}{0.5 \left(\frac{1.6F^{1/3}}{u_h} \right) x^{2/3} + h_s d} \quad (50)$$

Then,

$$H^{*2} = \left[\frac{c^2 x^{2d} (b+d)}{0.5 \left(\frac{1.6F^{1/3}}{u_h} \right) x^{2/3} + h_s d} \right]^2 \quad (51)$$

The maximum value for χ^* has the form:

In neutral condition, taking $d=0.95$ [8] χ^* becomes:

$$\chi^*(x, 0, 0, H) = [Q / (\pi \sigma_y \sigma_z u_h)] \exp \frac{-1}{2\sigma_z^2} \left[\frac{c^2 x^{2d} (b+d)}{0.28 \left(\frac{1.6F^{1/3}}{u_h} \right) x^{2/3} + h_s d} \right]^2 \quad (53)$$

In stable stability, there are two methods for maximum concentration at $z=0$ as follows:

(First)—In stable, where $d=0.67$ [8] the maximum value for χ^* is:

$$\chi^*(x, 0, 0, H) = [Q / (\pi \sigma_y \sigma_z u_h)] \exp \left[\frac{-c^2 x^{2d} (b+d)^2}{2d^2 h_s^2} \right] \quad (54)$$

(Second)—In stable (E and F), the stability parameter is written as follows:

$$s = \frac{g}{T_a} \left(\frac{\Delta \theta}{\Delta Z} \right) \quad (55)$$

where $\left(\frac{\Delta \theta}{\Delta Z} \right) = 0.02$ K/m for E and $\left(\frac{\Delta \theta}{\Delta Z} \right) = 0.035$ K/m for F [27].

Δh is written as follows:

$$\Delta h = 2.6 \left(\frac{F_b}{u_h s} \right)^{1/3} \quad (56)$$

H^* becomes:

$$H^* = \left(\frac{c^2 x^{2d} (b+d)}{d} \right)^{0.5} \quad (57)$$

Also, χ^* has the form:

$$\chi^*(x, 0, 0, H) = [Q / (\pi \sigma_y \sigma_z u_h)] \exp \left\{ \frac{-(b+d)}{2d} \right\} \quad (58)$$

8 Second case: H is a constant of downwind distance:

If H is constant, then, $\frac{\partial H}{\partial x} = 0$, assuming $H_c = h_s + 3w_o D/u$, where H_c is constant. H^* becomes:

$$H^* = \left[\frac{c^2 x^{2d} (b+d)}{d} \right]^{0.5} \quad (59)$$

The maximum value for χ^* has the form:

$$\chi^*(x, 0, 0; H) = \frac{Q}{\pi a c u_h x^{b+d}} \exp \left[\frac{-(b+d)}{2d} \right] \quad (60)$$

9 First case study

The decay distance and the concentration of pollutant at Inshas, Nuclear Research Center, Egyptian Atomic Energy Authority, were calculating. The stability was found in **stable** condition by using these data. The value of L was taken from Gifford work [13]. One uses Eq. (4) to calculate the values of u_s in terms of the wind velocity

$u(z)$ at 10 m height; hence, the values of the mixing height could be estimated from [2] for **stable** condition:

$$h = 0.4 \left(u_* \frac{L}{|f|} \right)^{1/2}, \text{ for } h/L > 0 \quad (61)$$

where f is the Coriolis parameter. v_d equals 0.04 m/s for Iodine [19]. Hence, the decay distance of iodine has been estimated using Eq. (23). The friction velocity and the mixing height with downwind distance are illustrated in Fig. 3. One gets the change of the friction velocity and mixing height from 0.25 to 0.3 m/s and 180 to 190 m, respectively [10].

The values of mixing height proportional with the decay distance are shown in Fig. 4; the normalized concentration (C/Q) of iodine at heights $z=0, 10, 50$ m as a function of downwind distance has been evaluated

by Eq. (22). One finds that the maximum normalized ground-level concentration equals 18×10^{-6} at 100 m and then decreases to reach near minimum value at 5000 m as shown in Fig. 5.

10 Second case study

C_o/Q is applied on the first research reactor at Inshas, Nuclear Research Center, Egyptian Atomic Energy Authority. h_s equals 43 m, D equals 1 m, w_o equals 4 m/s, and the total ventilation equals 39,965 m³/hr. Also, one takes $\alpha = -1$. Figure 6 shows that a straight-line fit well to these data in three conditions between C_o/Q and H .

11 Third case study

One gets the maximum concentration on the second research reactor at Inshas, Nuclear Research Center, Egyptian Atomic Energy Authority. h_s equals 27 m, D

Fig. 6 Shows that C_0/Q and H in three stabilities conditions

Fig. 7 The concentration for Cs-137 via downwind distance through different stabilities classes when " H " depends on " x "

equals 1 m, w_0 equals 0.3 m/s, and the total ventilation equals 3400 m³/hr [1]. One finds that the maximum concentrations are $6.2 \cdot 10^{-4}$, $4.8 \cdot 10^{-4}$ and $2 \cdot 10^{-4}$ Bq/m³ at 40 m in unstable, neutral and stable conditions, respectively; when plume rise depends on downwind distance as shown in Fig. 7, these values are less than the maximum system activity (MSA).

One finds that the effective height is quadratic with downwind distance in **unstable** case and the correlation coefficient (R) equals 0.99, ($x_{\max}$) equals 1000 m, and ($H_{\max}$) reaches 978 m. In **neutral** case, the curve is still quadratic and smaller than that in the unstable case, $R=1$. Also, in the **stable** case, the relation is linear,

$R=0.984$, $x_{\max}$ reaches to maximum value, and $H_{\max}$ will reach 59 m as shown in Fig. 8.

We find that the maximum concentrations are $2.5 \cdot 10^{-4}$, $2.1 \cdot 10^{-4}$ and $2 \cdot 10^{-4}$ Bq/m³ in **neutral**, **unstable** and **stable** conditions, respectively, at 40 m when H is a constant as shown in Fig. 9. Also, the relation between " H " and " x " is linear in **unstable** condition, $H_{\max}=163$ m when $x_{\max}=1000$ m and $R=0.997$. In **neutral** case, the relation is quadratic and $R=1$ and $H_{\max}=83$ m at $x_{\max}=1000$ m. Also, in **stable** case the relation is still quadratic and $R=0.998$, $H_{\max}=58.5$ m at $x_{\max}=1000$ m as shown in Fig. 10. We must take H that depends on x because one gets the values of " H " and the concentration near the true. This is the best result to reach it.

Fig. 8 The relation of "H" for Cs-137 via "x" when H depends on "x"

Fig. 9 The concentration for Cs 137 via downwind distance when H is a constant value

12 Conclusions

The concentration released from an elevated point source in the presence of both deposition and elevated mixed layer is estimated. The decay distance in the downwind distance is calculated. The maximum normalized concentration equals 18×10^{-6} (s/m^3) at 100 m and then decreases to reach minimum value at 5000 m at the ground level.

The logarithmic law of wind velocity in different conditions is used to get the plume rise, effective stack height and the normalized concentration at the axis of the

plume from the reactor release through different stability classes. We find that the ground-level concentration of iodine (I^{131}) agrees with the observed concentrations.

The values of extreme concentration are 6.2×10^{-4} , 4.8×10^{-4} and 2×10^{-4} Bq/m^3 at 40 m in unstable, neutral and stable conditions, respectively; when the plume rise depends on downwind distance and then decreases to values less than MSA, after that the values are close to zero at 400 m from the stack. In stable case, the maximum values of the ground-level concentration of air

Fig. 10 The variation of (H) for Cs-137 via (x) when H is a constant

pollutants were similar in the two cases. One obtains a good model when the plume rise depends on downwind distance and it is the best result.

Abbreviations

PBL: Planetary boundary layer is (PBL); x_{dn} : Decay distance of pollutant in neutral condition; X_{ds} : Decay distance of pollutant in the stable condition; X_{du} : Decay distance of pollutant in the unstable condition; D: Time dosage integral of the concentration; H: Effective stack height of the plume; C_0 : Concentration value at the edge of the plume; MSA: Maximum system activity.

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Competing interests

The authors declare that they have no competing interests in this section.

Author details

¹Mathematics and Theoretical Physics Department, NRC, Egyptian Atomic Energy Authority, Cairo, Egypt. ²Department of Physics, Faculty of Science, Beni-Suef University, Beni Suef, Egypt.

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