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Quaternionic quantum mechanics for $N=1$, 2, 4 supersymmetry

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Abstract

Background: Quaternions have emerged as powerful tools in higher-dimensional quantum mechanics as they provide homogeneous four-dimensional structure in quantum field theories, offer compact representations, and incorporate spin naturally. Quantum field theories then lead to the unification of fundamental interactions so the use of quaternion becomes necessary when we are dealing with higher-dimensional theories. On the other hand, supersymmetry is the theory of bosons and fermions and is an essential constituent of grand unified theories. The use of quaternion in supersymmetric field theories provides an excellent framework for higher-dimensional unification theories.

Result: A complete theory for supersymmetric quaternionic quantum mechanics has been constructed for N = 1, 2, 4 supersymmetry in terms of one, two, and four supercharges and Hamiltonians, respectively. It has been shown that $N=4$ SUSY is the quaternionic extension of the $N=2$ complex SUSY and $N=1$ real SUSY; also spin is the natural outcome of using quaternion units. Pauli and Dirac Hamiltonian and their relationship have also been obtained in quaternion space. It has been shown that quaternionic quantum mechanics are superior to ordinary and complex quantum mechanics because in the quaternion framework we do not need three different theories for $N=1,2,4$ SQM but a single theory only.

Conclusions: It has been concluded that $N=1$ real SUSY is equal to $N=2$ complex SUSY which in turn is equal to $N=4$ quaternion SUSY so one can arrive at higher-dimensional quantum field theories starting from lower-dimensional quantum theories. Higher-dimensional quaternion field theories are suitable for nonphotonic light cone particles which are not allowed in complex QFT, also noncommutative nature of quaternion gives an extra degree of freedom and may provide the possibility of some new particle, dark matter, or new phenomenon. Though quaternions provide an excellent framework in higher-dimensional field theories, there are certain challenges due to their noncommutativity as calculations become tedious where large terms are involved. Keeping in view the noble features of quaternion, we expect some development to get a better understanding of $N=8$ supergravity, maximal supergravity ($D=11-n$), and maximal supersymmetry theories ($N=10$) in terms of quaternion operators.

Keywords: Supersymmetry, Quaternion, Relativistic quantum mechanics, Quantum field theories

1 Background

In recent years, quaternions have emerged as powerful tools in higher-dimensional quantum mechanics as they provide homogeneous four-dimensional structures in relativistic quantum mechanics and provide representations in terms of compact notations [1–3]. Also, spin is a natural outcome of using quaternion as they are represented in terms of Pauli spin matrices [4, 5] so their use becomes necessary while dealing with nonzero spin particles. Quaternion product is noncommutative so we get an extra degree of freedom in expressions which can lead to some new phenomenon, particles, and explanations of some undefined questions in particle physics [6,

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7]. Higher-dimensional quaternionic quantum field theories are suitable for nonphotonic light cone particles and massive Higgs bosons which are not allowed in complex field theories [7].

Quaternions are hypercomplex numbers that are similar to complex numbers in values but noncommutative. Quaternions were extensively used in quantum mechanics by S.L. Adler [8, 9]. Recently, quaternions are gaining much popularity since their use in relativistic quantum mechanics by S. Giardino [10], B.C. Chanyal [11] and in quantum field theory was used by B.C. Chanyal [12], S.D. Leo [13, 14], S. Ulrich [15] and H. Sobhani et al. [16–18].

Supersymmetric theories were first appeared in the field theories and are exact symmetric to explain bosonic and fermionic fields in a single theory and serve as the basic outline to unite all fundamental interactions [16]. Supersymmetry is the theory of bosons and fermions and is an essential constituent of grand unified theories of four fundamental interactions [19, 20]. Quaternion in SUSY quantum mechanics was first used by A.J Davis [21]. Several authors studied higher-dimensional SUSY quantum mechanics in detail [22–27].

Keeping in view the utility of quaternion and supersymmetry in quantum mechanics, we have tried to construct a single theory in terms of quaternionic operators for $N=1,2,4$ supersymmetric quantum mechanics. $N=1$ SUSY quantum mechanics is explained in terms of one supercharge. We have obtained expressions for supercharges, Pauli Hamiltonian, and Dirac Hamiltonian. $N=2$ SUSY quaternionic quantum mechanics has been constructed in terms of two complex supercharges and Hamiltonians; it contains an additional spin 1/2 degree of freedom [21]. We also observed that there should be a vanishing ground state for unbroken SUSY. By replacing complex numbers with quaternion units, $N=2$ SUSY is extended to $N=4$ SUSY and four supersymmetric generators have been constructed as discussed by Junker [26] Hull [27]. It is concluded that quaternionic quantum mechanics.

are superior to ordinary and complex quantum mechanics as we need only a single theory instead of three different theories for $N=1,2,4$ SQM and is proved as the best theory for higher-dimensional quantum mechanics as it automatically provides a four-dimensional structure and spin to it.

2 Method

2.1 $N=1, 2, 4$ supersymmetric quantum mechanics

Let us analyze a quantum system, defined by a Hamiltonian $\hat{H}$ (Hermitian in nature) acting on some Hilbert space which is constructed in terms of N self-adjoint operators $\hat{M}_i = \hat{M}_i^{\dagger}$. Such quantum system is called supersymmetric provided the following anti-commutation relation is valid.

$$\left\{ \hat{M}_i, \hat{M}_j \right\} = \hat{H}\delta_{ij}, \quad (i, j = 0, 1, 2, 3, 4, \ldots\ldots, N) \tag{1}$$

where curly bracket represents anticommuting relation. The self-adjoint operators $\hat{M}_i$ & $\hat{M}_j$ are called supercharges, and the Hamiltonian $\hat{H}$ is said to be SUSY Hamiltonian.

2.2 $N=1$ quaternionic supersymmetric quantum mechanics

$N=1$ SQM is defined in terms of only one supercharge called a generator of $N=1$ SUSY. One-dimensional supersymmetric quantum mechanics is then described by the graded algebra and can be expressed as

$$\begin{aligned} \left\langle \psi \middle| \hat{H} \middle| \psi \right\rangle &= \left\langle \psi \middle| \hat{M}^{\dagger}\hat{M} \middle| \psi \right\rangle + \left\langle \psi \middle| \hat{M}\hat{M}^{\dagger} \middle| \psi \right\rangle \\ &= \left| \hat{M} \middle| \psi > \right|^2 + \left| \hat{M}^{\dagger} \middle| \psi > \right|^2 \geq 0 \end{aligned} \tag{2}$$

Here $\left| \hat{H} \right| \psi >$ is the corresponding eigenstate. We may extend $N=1$ quaternionic SUSY to the relativistic quantum mechanics where a system [22] is defined by a Pauli Hamiltonian for a spin 1/2 particle in an external magnetic field. Let us consider two quaternion gauge potentials $\vec{A}_\mu(x,t)$ and $\vec{B}_\mu(x,t)$. The two external quaternion gauge magnetic fields are given as

$$C = \vec{\nabla} \times \vec{A}_\mu(x,t) \text{ and } C' = \vec{\nabla} \times \vec{B}_\mu(x,t) \tag{3}$$

where $\vec{A}_\mu(x,t)$ and $\vec{B}_\mu(x,t)$ are quaternion potentials ($\mu = 0, 1, 2, 3$) and defined as

$$\vec{A}_\mu(x,t) = A_0 + \sum_{l=1}^{3} e_l \vec{A}_l \text{ and } \vec{B}_\mu(x,t) = B_0 + \sum_{l=1}^{3} e_l \vec{B}_l \tag{4}$$

Here A_0 and B_0 are the scalar part of electric and magnetic field, respectively, and $\vec{A}_l, \vec{B}_l$ are vector part of the corresponding electric and magnetic field. We may introduce self-adjoint supercharge ($\hat{M}_D$) in electromagnetic field system in terms of momentum ($\vec{p}_l$), electric ($\vec{A}_l$), and magnetic field ($\vec{B}_l$), i.e.,

$$\hat{M}_D = ie_l\left(\vec{p}_l - ie_l\vec{A}_l + ie_l\vec{B}_l\right) = \hat{M}_D^{\dagger} \tag{5}$$

in the above equation, if we substitute the scalar part of quaternion i. e. $e_l \to e_0 \to 1$, we get only one supercharge which is the generator of $N=1$ SUSY; henceforth, we get Pauli Hamiltonian [21]

$$\hat{H}_P = 2\hat{M}_D^2 = 2\left\{ ie_l\left(\vec{p}_l - ie_l\vec{A}_l + ie_l\vec{B}_l\right) \right\}^2 \tag{6}$$

which is described as the Pauli Hamiltonian for a spin $-1/2$ particles. Accordingly, we may write Dirac Hamiltonian $\widehat{H}_D$ as

$$\begin{aligned}\hat{H}_D &= \sum_{l=1}^{3}\alpha_l\left(\vec{p}_l - ie_l\vec{A}_l + ie_l\vec{B}_l\right) + \beta m \\ &= \begin{bmatrix} m & ie_l\left(\vec{p}_l - ie_l\vec{A}_l + ie_l\vec{B}_l\right) \\ ie_l\left(\vec{p}_l - ie_l\vec{A}_l + ie_l\vec{B}_l\right) & -m \end{bmatrix} \\ &= \begin{bmatrix} m & \hat{M}_D^\dagger \\ \hat{M}_D & -m \end{bmatrix}\end{aligned} \tag{7}$$

; similarly Hamiltonian for $N=1$ SUSY is obtained by substituting $e_l \to e_0 \to 1$; here we can replace Dirac matrices (α_l, β) with quaternions (e_l) as

$$\alpha_l = \begin{bmatrix} 0 & ie_l \\ ie_l & 0 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{8}$$

Squaring the Dirac Hamiltonian, we get

$$\begin{aligned}\hat{H}_D^2 &= \begin{bmatrix} \hat{M}_D\hat{M}_D^\dagger + m^2 & 0 \\ 0 & \hat{M}_D\hat{M}_D^\dagger + m^2 \end{bmatrix} \\ &= \begin{bmatrix} \hat{M}_D^2 + m^2 & 0 \\ 0 & \hat{M}_D^2 + m^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\hat{H}_P}{2} + m^2 & 0 \\ 0 & \frac{\hat{H}_P}{2} + m^2 \end{bmatrix}\end{aligned} \tag{9}$$

where $\hat{H}_P$ is Pauli's Hamiltonian; we also get the relationship for Dirac and Pauli Hamiltonian

$$\left[\hat{M}_D, \hat{H}_D\right] = 0, \quad \left[\hat{M}_D, \hat{H}_P\right] = 0, \quad \left[\hat{M}_D, \hat{M}_D\right] = \hat{H}_P \tag{10}$$

2.3 $N=2$ quaternionic supersymmetric quantum mechanics

$N=2$ SUSY was discussed by Witten [16] as a simple model, which consists of an additional spin $-1/2$ degree of freedom, and accordingly, we may write two supercharges $\hat{M}_1 \& \hat{M}_2$ in terms of quaternion units e_1 and e_2 and potential $\emptyset(x)$ as follows

$$\begin{aligned}\hat{M}_1 &= \frac{1}{\sqrt{2}}(p \times ie_1 + \emptyset(x) \times ie_2) \;\&\; \hat{M}_2 \\ &= \frac{1}{\sqrt{2}}(p \times ie_2 - \emptyset(x) \times ie_1)\end{aligned} \tag{11}$$

Then the Hamiltonian is given by

$$\hat{H} = 2\hat{M}_1^2 = 2\hat{M}_2^2 = p^2 + \emptyset^2 - 2e_3p\emptyset = p^2 + \emptyset^2 - 2i\sigma_3p\left({}^{-\mathrm{d}}/_{\mathrm{d}x}\right)\emptyset = p^2 + \emptyset^2 - 2\sigma_3\emptyset' \tag{12}$$

here σ_3 is the Pauli spin matrix related to the quaternion unit as $\sigma_3 = ie_3$. Here we can see that spin is the natural outcome of using quaternion units. Now Hamiltonian can be written in terms of super partner Hamiltonians $\hat{H}_+$ and $\hat{H}_-$ as

$$\hat{H} = \begin{bmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{bmatrix} = \begin{bmatrix} p^2 + \emptyset^2 + 2\emptyset' & 0 \\ 0 & p^2 + \emptyset^2 - 2\emptyset' \end{bmatrix} \tag{13}$$

For $N=2$ SUSY, two complex supercharges $\hat{M}_1$ & $\hat{M}_2$ are formed by substituting quaternion units by complex unit—i and Hamiltonian $\hat{H}$ satisfying the following relations

$$\begin{aligned}\hat{M}_1\hat{M}_2 &= -\hat{M}_2\hat{M}_1 = -\frac{1}{2}\left(ip^2 + i\emptyset^2 + p\emptyset\right) \;\&\; \hat{H} = 2\hat{M}_1^2 \\ &= 2\hat{M}_2^2 = p^2 + \emptyset^2 - 2ip\emptyset = p^2 + \emptyset^2 - 2\emptyset'\end{aligned} \tag{14}$$

Let the two complex supercharges are given by

$$\hat{M} = \frac{1}{\sqrt{2}}\left(\hat{M}_1 + i\hat{M}_2\right) \;\&\; \hat{M}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{M}_1 - i\hat{M}_2\right) \tag{15}$$

where i is a complex quantity and belongs to c $(1, i)$ space. These supercharge $\hat{M}, \hat{M}^\dagger$ and Hamiltonian $\hat{H}$ should satisfy the SUSY algebra,

$$\hat{M}^2 = \hat{M}^\dagger = 0 \;\&\; \hat{H} = \left\{\hat{M}, \hat{M}^\dagger\right\}$$

Now to satisfy the relation $\hat{M}^2 = \hat{M}^{\dagger 2} = 0$, the complex supercharges are given by nilpotent matrices, which are defined as follows

$$\hat{M} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \;\&\; \hat{M}^\dagger = \begin{bmatrix} 0 & 0 \\ a^\dagger & 0 \end{bmatrix} \tag{16}$$

Hence the Hamiltonian becomes

$$\hat{H} = \left\{\hat{M}, \hat{M}^\dagger\right\} = \begin{bmatrix} aa^\dagger & 0 \\ 0 & a^\dagger a \end{bmatrix} \tag{17}$$

where a& $a^\dagger$ are annihilation and creation operators defined as

$$a = \frac{1}{\sqrt{2\omega}}(-ip + \omega q) \;\&\; a^\dagger = \frac{1}{\sqrt{2\omega}}(ip + \omega q) \tag{18}$$

Substituting $2\omega = 1$ and $\omega q = U(x)$, $U(x)$ is real super potential; we can write a and $a^\dagger$ in the following manner

$$a = (-ip + U(x)) = -\frac{\mathrm{d}}{\mathrm{d}x} + U \text{ and } a^{\dagger} = (ip + U(x)) = \frac{\mathrm{d}}{\mathrm{d}x} + U \tag{19}$$

The supercharges for this case are given by

$$\hat{M} = \begin{bmatrix} 0 & -\frac{d}{dx} + U \\ 0 & 0 \end{bmatrix} \;\&\; \hat{M}^{\dagger} = \begin{bmatrix} 0 & 0 \\ \frac{\mathrm{d}}{\mathrm{d}x} + U & 0 \end{bmatrix} \tag{20}$$

and the Hamiltonian is given by

$$\hat{H} = \begin{bmatrix} \hat{H}_{+} & 0 \\ 0 & \hat{H}_{-} \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - U' + U^2 & 0 \\ 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U' + U^2 \end{bmatrix} \tag{21}$$

Here $\hat{H}$, $\hat{M}$, and $\hat{M}^{\dagger}$ satisfy the SUSY algebra given by

$$\left[\hat{H}, \hat{M}\right] = \left[\hat{H}, \hat{M}^{\dagger}\right] = 0, \;\; \{\hat{M}, \hat{M}\} = \left\{\hat{M}^{\dagger}, \hat{M}^{\dagger}\right\} = 0 \;\&\; \hat{H} = \left\{\hat{M}, \hat{M}^{\dagger}\right\} \tag{22}$$

where square bracket represents commutation relation while curly bracket represents anticommutation relation. We consider ψ as a two-component (ψ_a & ψ_b) spinor given by

$$\psi = \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} \tag{23}$$

and necessary condition for SUSY to be a good supersymmetry that supercharges annihilate the ground state

$$\hat{M}\Big|\psi >= \hat{M}^{\dagger}\Big|\psi >= 0 \tag{24}$$

In terms of energy, the condition for SUSY to be a good SUSY is that the ground state energy should be zero. Using Eqs. (20) and (23), we get

$$\hat{H} = \begin{bmatrix} \left(-\vec{e}_l.\vec{p}_l + U\right)\left(-\vec{e}_l.\vec{p}_l + U\right) & 0 \\ 0 & \left(-\vec{e}_l.\vec{p}_l + U\right)\left(-\vec{e}_l.\vec{p}_l + U\right) \end{bmatrix} \tag{32}$$

$$-\psi_b' + U\psi_b = 0 \text{ and } \psi_a' + U\psi_a = 0 \tag{25}$$

where U is given by

$$U = \frac{\psi_b'}{\psi_b} = -\frac{\psi_a'}{\psi_a};$$

So that

$$\psi_b = e^{-\int_{x_0}^{x} \vec{U}(s).\mathrm{d}\vec{S}} \text{ and } \psi_a = e^{-\int_{x_0}^{x} \vec{U}(s).\mathrm{d}\vec{S}}. \tag{26}$$

Let us define the quaternion wave function as two-component complex spinors

$$\psi = \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} = \begin{bmatrix} \psi^{+} \\ \psi^{-} \end{bmatrix} \tag{27}$$

2.4 $N=4$ supersymmetric quantum mechanics

According to Hull [22], $N=4$ SUSY QM can be formed from $N=2$ SUSY QM by extending the complex number i to three imaginary units, described as quaternion units. Thus $N=4$ SUSY QM can be obtained by replacing i by $\vec{e}$ in Eqs. (18) and (19). Then we get

$$a = \left(-\vec{e}_l.\vec{p}_l + U\right) \text{ and } a^{\dagger} = \left(-\vec{e}_l.\vec{p}_l + U\right) \tag{28}$$

Here we consider U as the quaternionic super potential by taking only the imaginary part and leaving the real part and is defined by

$$U = \sum_{l=1}^{3} e_l w_l = e_1 w_1 + e_2 w_2 + e_3 w_3 \tag{29}$$

$$U^{\dagger} = -\sum_{l=1}^{3} e_l w_l = -e_1 w_1 - e_2 w_2 - e_3 w_3 \tag{30}$$

Hence the supercharges for this case are deduced as

$$\hat{M} = \begin{bmatrix} 0 & \left(-\vec{e}_l.\vec{p}_l + U\right) \\ 0 & 0 \end{bmatrix} \;\&\; \hat{M}^{\dagger} = \begin{bmatrix} 0 & 0 \\ \left(-\vec{e}_l.\vec{p}_l + U^{\dagger}\right) & 0 \end{bmatrix} \tag{31}$$

if we replace $\vec{e}_l \rightarrow \vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3$ we obtain 4 supercharges for $N=4$ SQM called generators of $N=4$ SQM.

and the Hamiltonian becomes

Similarly, we get four super-partner Hamiltonians for $N=4$ SQM by replacing $\vec{e}_l \rightarrow \vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3$ and satisfy the usual SUSY algebra given by Eq. (22). Equation (32) reduces to the following expression of supercharges on using the value of U from Eqs. (30) and (31), i.e.,

$$\hat{M} = \begin{bmatrix} 0 & \vec{e}_l\left(-\vec{p}_l + w_l\right) \\ 0 & 0 \end{bmatrix} \;\&\; \hat{M}^{\dagger} = \begin{bmatrix} 0 & 0 \\ -\vec{e}_l\left(\vec{p}_l - w_l\right) & 0 \end{bmatrix} \tag{33}$$

Corresponding wave function is given by

$$\psi = \psi_0 + e_1\psi_1 + e_2\psi_2 + e_3\psi_3$$

Or

$$\psi = \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} = \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix} = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

where ψ^+ and ψ^- are again two-component spinors corresponding to an upper and lower component of Dirac spinor in the following manner

$$\psi^+ = \psi_0 + e_1\psi_1 = \begin{bmatrix} \phi^+ \\ 0 \end{bmatrix} \text{ and } \psi^- = \psi_2 - e_1\psi_3 = \begin{bmatrix} 0 \\ \phi^- \end{bmatrix} \tag{34}$$

where we have used $a = \left(-\vec{e}_l.\vec{p}_l + U\right)$. If we substitute quaternion basis elements e_l by $e_l = -i\sigma_l$, so that $a = -i\sigma_l\left(-\vec{p}_l + w_l\right)$, here σ_l is the Pauli spin matrix. Showing that by using quaternionic algebra spin automatically appears in the structure. Hence spin naturally occurs in quaternionic quantum mechanics, which is not possible in $N=2$ supersymmetric quantum mechanics. Replacing $e_l = -i\sigma_l$, we get the following representation for Dirac matrices as follows

$$\gamma_l = \begin{bmatrix} e_l & 0 \\ 0 & e_l \end{bmatrix} \tag{35}$$

where

$$\gamma_l^\dagger = -\gamma_l, \ \ tr\gamma_l = 0 \ \& \ \gamma_l\gamma_k + \gamma_k\gamma_l = -2\delta_{lk} \tag{36}$$

These are the matrices, which have been used by Rotelli [20] in formulating the quaternionic Dirac equation.

3 Result

We have formulated $N=1$ SUSY quantum mechanics in the presence of electromagnetic field which is defined in terms of one supercharge by considering only the scalar quaternion part. The relation between Pauli and Dirac Hamiltonian has also been obtained, which gives the relationship between ordinary and relativistic quantum mechanics. The self-adjoint supercharges, Dirac Hamiltonian, and Pauli Hamiltonian satisfy SUSY algebra and are consistent with supersymmetric theory. It has been shown that $N=2$ SUSY is defined in terms of two complex supercharges and consists of an additional spin 1/2 degrees of freedom, and accordingly supercharges, corresponding Dirac Hamiltonian and Pauli Hamiltonian in terms of quaternion units, are discussed. The complex supercharges and Hamiltonian are also discussed by replacing quaternion units with complex no (i) and the relation between them has been established.

We have also developed a complete theory for $N=4$ SUSY quantum mechanics and constructed the SUSY generators in the same manner as discussed by Hull [22]. We have obtained four supercharges by taking scalar and vector parts of the quaternion. We also obtained Dirac matrices and their relationships as discussed by Rotelli [20]. It has been shown that by using the quaternion framework we do not need three different theories for $N=1,2,4$ SQM but a single theory only.

4 Discussion

We have developed a complete theory for $N=1,2,4$ supersymmetry, and we can develop higher-dimensional theories from lower-dimensional by using quaternion units; also we have obtained a relationship between Pauli (nonrelativistic quantum mechanics) and Dirac Hamiltonian (relativistic quantum mechanics) so by using this theory we can formulate relativistic theory with the help of nonrelativistic or vice versa. In this theory, we have seen that only a single theory is required for $N=1, 2, 4$ SQM instead of three different theories. We have seen that quaternion representation is compact and can provide an excellent framework for higher-dimensional quantum mechanics such as $N=8$ supergravity, maximal supergravity($D=11-n$), and maximal supersymmetry ($N=10$) theories that can be developed by taking three quaternions (Euclidean) spaces; also spin is the natural outcome of quaternions as they can be represented by Pauli spin matrices so quaternion should be necessarily used in theories for a particle having spin-1/2. Since it contains noncommutative four-dimensional space, we get an extra degree of freedom in different interactions and this can lead to some new phenomena, particles, and explanations of some undefined questions in theoretical physics. It has been shown that the SUSY will be good supersymmetry unless and until we impose the necessary condition showing that ground state energy must be vanishing but at the same time it is difficult to develop quaternion field theories where several parameters are used and it becomes difficult for field theories to remain consistent with laws of quantum mechanics due to noncommutative nature of quaternion units.

5 Conclusions

In this article, we have developed a complete theory for $N=1, 2, 4$ SUSY quantum mechanics. It has been concluded that $N=1$ real SUSY is equal to $N=2$ complex SUSY which in turn is equal to $N=4$ quaternion SUSY Q. M. The relationship between nonrelativistic and relativistic quantum mechanics has also been used in terms of Pauli and Dirac's Hamiltonians using quaternion operators. With the help of this article, one can arrive at higher-dimensional quantum field theories starting from lower-dimensional quantum theories. Higher-dimensional quaternion field theories are suitable for nonphotonic light cone particles and massive particles which are not allowed in complex quantum field theories; also noncommutative nature of quaternion gives an extra degree of freedom and may provide the possibility of some new particle, dark matter, or new phenomenon.

Though quaternions provide an excellent framework for higher-dimensional quantum field theories due to their uniqueness in providing four-component homogeneous space–time, compact notations, and spin inheritance, there are certain challenges due to their four-component structure and noncommutativity as calculations become tedious where large terms are included but this problem can be overcome by developing programming techniques.

Keeping in view the Nobel features of quaternions in higher-dimensional field theories, we expect some development to get a better understanding of $N=8$ supergravity, maximal supergravity$(D=11-n)$, and maximal supersymmetry theories $(N=10)$ in terms of quaternion operators.

Abbreviations
SUSY: Supersymmetry; QFT: Quantum field theory; SQM: Supersymmetric quantum mechanics.

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References

1. Rawat S, Negi OPS (2009) Quaternion Dirac equation and supersymmetry. Int J Theor Phys 48(8):2222–2234. https://doi.org/10.1007/s10773-009-0003-4
2. Rawat AS, Rawat S, Li T, Negi OPS (2012) Supersymmetrization of quaternion Dirac equation for generalized fields of dyons. Int J Theor Phys 51(10):3274–3289. https://doi.org/10.1007/s10773-012-1206-7
3. De Leo S, Giardino S (2014) Dirac solutions for quaternionic potentials. J Math Phys 55(2):022301. https://doi.org/10.1063/1.4863903
4. Jolly DC (1984) Isomorphic matrix representation of quaternion field theories. Lett Nuovo Cim 39(9):185
5. Silveria AD (1985) Lett Nuovo Cim 44:80
6. Rawat S, Negi OPS (2009) Quaternionic formulation of supersymmetric quantum mechanics. Int J Theor Phys 48(2):305–314. https://doi.org/10.1007/s10773-008-9803-1
7. Giardino S (2021) Quaternionic Klein Gordon equation. arXiv:2105.11270v1(quant-ph)
8. Adler SL (1995) Quaternionic quantum mechanics. Oxford University Press, Oxford
9. Adler SL (1986) Quaternionic quantum field theory. Commun Math Phys 104:611–656
10. Giardino S (2016) Quaternionic particle in relativistic box. Found Phys 46(4):473–483
11. Chanyal BC (2017) Generalized Klein–Gordon field equations with octonion space-time (OST) algebra. Chin J Phys 55(2):432–443. https://doi.org/10.1016/j.cjph.2017.01.003
12. Chanyal BC (2017) A relativistic quantum theory of dyons wave propagation. Can J Phys 95(12):1200–1207. https://doi.org/10.1139/cjp-2017-0080
13. De Leo S, Ducati G, Giardino S (2015) Quaternionic dirac scattering. J Phys Math 6:1000130
14. De Leo S, Ducati G, Leonardi V, Pereira K (2010) A closed formula for the barrier transmission coefficient in quaternionic quantum mechanics. J Math Phys 51(11):113504. https://doi.org/10.1063/1.3504165
15. Ulrych S (2013) Higher spin quaternion waves in the Klein–Gordon theory. Int J Theor Phys 52(1):279–292. https://doi.org/10.1007/s10773-012-1330-4
16. Sobhani H, Hassanabadi H (2016) Scattering in quantum mechanics under quaternionic Dirac delta potential. Can J Phys 94(3):262–266
17. Sobhani H, Hassanabadi H, Chung WS (2017) Observations of the Ramsauer-Townsend effect in quaternionic quantum mechanics. Eur Phys J C 77(6):425
18. Sobhani H, Hassanabadi H (2017) New face of Ramsauer–Townsend effect by using a Quaternionic double Dirac potential. Indian J Phys 91(10):1205–1209
19. Witten E (1981) Dynamical breaking of supersymmetry. Nucl Phys B 188(3):513–554
20. Rotelli P (1989) Quaternion trace theorems and first order electron-muon scattering. Mod Phys Lett A 4(18):1763–1771
21. Davies AJ (1994) Supersymmetric quaternionic quantum mechanics. Phys Rev A 49(2):714
22. Das A, Okubo S, Pernice SA (1997) Higher-dimensional SUSY quantum mechanics. Mod Phys Lett A 12:581
23. Nauta L (2009) Supersymmetric quantum mechanics: an introduction for undergraduates. Bachelor Project Physics and Astronomy, Institute of Theoretical Physics Univ. of Amsterdam
24. Nygren E (2010) Supersymmetric quantum mechanics. Bachelor Thesis at the Institute for Theoretical Physics, Science Faculty, University of Bern
25. Sukumar CV (1985) Supersymmetry and the Dirac equation for a central Coulomb field. J Phys A Math Gen 18:L697
26. Junker G (1996) Supersymmetric methods in quantum mechanics and statistical physics. Springer, Berlin
27. Hull CM The geometry of supersymmetric quantum mechanics. arXiv: hep-the/9910028

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